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**Calculus of Variations**

# A threshold phenomenon for embeddings of $H_0^m$ into Orlicz spaces

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**Abstract** Given an open bounded domain $\Omega \subset \mathbb{R}^{2m}$ with smooth boundary, we consider a sequence $(u_k)_{k\in\mathbb{N}}$ of positive smooth solutions to

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} & \text{in } \Omega \\ u_k = \partial_\nu u_k = \cdots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda_k \to 0^+$. Assuming that the sequence is bounded in $H_0^m(\Omega)$, we study its blow-up behavior. We show that if the sequence is not precompact, then

$$\liminf_{k\to\infty} \|u_k\|_{H_0^m}^2 := \liminf_{k\to\infty} \int_\Omega u_k(-\Delta)^m u_k dx \geq \Lambda_1,$$

where $\Lambda_1 = (2m-1)!\mathrm{vol}(S^{2m})$ is the total $Q$-curvature of $S^{2m}$.

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## 1 Introduction and statement of the main result

Let $\Omega \subset \mathbb{R}^{2m}$ be open, bounded and with smooth boundary, and let a sequence $\lambda_k \to 0^+$ be given. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ of smooth solutions to

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} & \text{in } \Omega \\ u_k > 0 & \text{in } \Omega \\ u_k = \partial_\nu u_k = \cdots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

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Assume also that

$$\int_\Omega u_k(-\Delta)^m u_k dx = \lambda_k \int_\Omega u_k^2 e^{mu_k^2} dx \to \Lambda \geq 0 \quad \text{as } k \to \infty. \tag{2}$$

In this paper we shall prove

**Theorem 1** *Let $(u_k)$ be a sequence of solutions to (1), (2). Then either*

(i) $\Lambda = 0$ *and* $u_k \to 0$ *in* $C^{2m-1,\alpha}(\Omega)$,[1] *or*
(ii) *We have* $\sup_\Omega u_k \to \infty$ *as* $k \to \infty$. *Moreover there exists* $I \in \mathbb{N}\backslash\{0\}$ *such that* $\Lambda \geq I\Lambda_1$, *where* $\Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$, *and up to a subsequence there are* $I$ *converging sequences of points* $x_{i,k} \to x^{(i)}$ *and of positive numbers* $r_{i,k} \to 0$, *the latter defined by*

$$\lambda_k r_{i,k}^{2m} u_k^2(x_{i,k}) e^{mu_k^2(x_{i,k})} = 2^{2m}(2m-1)!, \tag{3}$$

*such that the following is true:*

1. *For every* $1 \leq i \leq I$ *we have* $\lim_{k\to\infty} \frac{\mathrm{dist}(x_{i,k},\partial\Omega)}{r_{i,k}} = +\infty$.
2. *If we define*

$$\eta_{i,k}(x) := u_k(x_{i,k})(u_k(x_{i,k} + r_{i,k}x) - u_k(x_{i,k})) + \log 2$$

*for* $1 \leq i \leq I$, *then*

$$\eta_{i,k}(x) \to \eta_0(x) = \log \frac{2}{1+|x|^2} \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}) \quad (k \to \infty). \tag{4}$$

3. *For every* $1 \leq i \neq j \leq I$ *we have* $\lim_{k\to\infty} \frac{|x_{i,k}-x_{j,k}|}{r_{i,k}} = \infty$.
4. *Set* $R_k(x) := \inf_{1\leq i\leq I} |x - x_{i,k}|$. *Then*

$$\lambda_k R_k^{2m}(x) u_k^2(x) e^{mu_k^2(x)} \leq C, \tag{5}$$

*where* $C$ *does not depend on* $x$ *or* $k$.

*Finally* $u_k \rightharpoonup 0$ *in* $H^m(\Omega)$ *and* $u_k \to 0$ *in* $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash\{x^{(1)}, \ldots, x^{(I)}\})$.

Solutions to (1) arise from the Adams–Moser–Trudinger inequality [1] (see also [9,18, 29]):

$$\sup_{u\in H_0^m(\Omega),\, \|u\|^2_{H_0^m} \leq \Lambda_1} \fint_\Omega e^{mu^2} dx = c_0(m) < +\infty, \tag{6}$$

where $c_0(m)$ is a dimensional constant, and $H_0^m(\Omega)$ is the Beppo–Levi defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm[2]

$$\|u\|_{H_0^m} := \|\Delta^{\frac{m}{2}} u\|_{L^2} = \left(\int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx\right)^{\frac{1}{2}}, \tag{7}$$

[1] Here and in the following $\alpha \in [0, 1)$ is an arbitrary Hölder exponent.

[2] The norm in (7) is equivalent to the usual Sobolev norm $\|u\|_{H^m} := \left(\sum_{\ell=0}^m \|\nabla^\ell u\|_{L^2}\right)^{\frac{1}{2}}$, thanks to elliptic estimates.

and we used the following notation:

$$\Delta^{\frac{m}{2}} u := \begin{cases} \Delta^n u \in \mathbb{R} & \text{if } m = 2n \text{ is even,} \\ \nabla \Delta^n u \in \mathbb{R}^{2m} & \text{if } m = 2n+1 \text{ is odd.} \end{cases} \tag{8}$$

In fact (1) is the Euler-Lagrange equation of the functional

$$F(u) := \frac{1}{2} \int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx - \frac{\lambda}{2m} \int_\Omega e^{mu^2} dx$$

(where $\lambda = \lambda_k$ plays the role of a Lagrange multiplier), which is well defined and smooth thanks to (6), but does not satisfy the Palais–Smale condition. For a more detailed discussion, in the context of Orlicz spaces, we refer to [26].

The function $\eta_0$ which appears in (4) is a solution of the higher-order Liouville's equation

$$(-\Delta)^m \eta_0 = (2m-1)! e^{2m\eta_0}, \quad \text{on } \mathbb{R}^{2m}. \tag{9}$$

We recall (see e.g. [16]) that if $u$ solves $(-\Delta)^m u = V e^{2mu}$ on $\mathbb{R}^{2m}$, then the conformal metric $g_u := e^{2u} g_{\mathbb{R}^{2m}}$ has $Q$-curvature $V$, where $g_{\mathbb{R}^{2m}}$ denotes the Euclidean metric. This shows a surprising relation between Eq. (1) and the problem of prescribing the $Q$-curvature. In fact $\eta_0$ has also a remarkable geometric interpretation: If $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection, then

$$e^{2\eta_0} g_{\mathbb{R}^{2m}} = (\pi^{-1})^* g_{S^{2m}}, \tag{10}$$

where $g_{S^{2m}}$ is the round metric on $S^{2m}$. Then (10) implies

$$(2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx = \int_{S^{2m}} Q_{S^{2m}} \mathrm{dvol}_{g_{S^{2m}}} = (2m-1)! |S^{2m}| = \Lambda_1. \tag{11}$$

This is the reason why $\Lambda \geq I\Lambda_1$ in case (ii) of Theorem 1 above, compare Proposition 7.

Theorem 1 has been proven by Adimurthi and Struwe [3] and Adimurthi and Druet [2] in the case $m = 1$, and by Robert and Struwe [22] for $m = 2$. The extraction of a blow-up profile from a concentrating sequence of solutions to a nonlinear PDE was pioneered by Sack and Uhlenbeck [23] and Wente [30]. Their ideas were later expanded in various ways by Struwe [24,25], Brezis and Coron [6,7] who, in particular, first wrote down separation conditions like conditions 1 and 3 in part (ii) of Theorem 1 (see also the works of Parker [20], Hebey and Robert [13] and many others). For further motivations and references we refer to Struwe [28]. Here, instead, we want to point out the main ingredients of our approach. Crucial to the proof of Theorem 1 are the gradient estimates in Lemma 6 and the blow-up procedure of Proposition 7. For the latter, we rely on a concentration-compactness result from [17] and a classification result from [16], which imply, together with the gradient estimates, that at the finitely many concentration points $\{x^{(1)}, \ldots, x^{(I)}\}$, the profile of $u_k$ is $\eta_0$, hence an energy not less that $\Lambda_1$ accumulates, namely

$$\lim_{R\to 0} \limsup_{k\to\infty} \int_{B_R(x^{(i)})} \lambda_k u_k^2 e^{mu_k^2} dx \geq \Lambda_1, \quad \text{for every } 1 \leq i \leq I.$$

As for the gradient estimates, if one uses (1) and (2) to infer $\|\Delta^m u_k\|_{L^1(\Omega)} \leq C$, then elliptic regularity gives $\|\nabla^\ell u_k\|_{L^p(\Omega)} \leq C(p)$ for every $p \in [1, 2m/\ell)$. These bounds, though, turn out to be too weak for Lemma 6 (see also the remark after Lemma 5). One has, instead, to fully exploit the integrability of $\Delta^m u_k$ given by (2), namely $\|\Delta^m u_k\|_{L(\log L)^{1/2}(\Omega)} \leq C$, where $L(\log L)^{1/2} \subsetneq L^1$ is the Zygmund space. Then an interpolation result from [5] gives uniform estimates for $\nabla^\ell u_k$ in the Lorentz space $L^{(2m/\ell,2)}(\Omega)$, $1 \le \ell \le 2m-1$, which are sharp for our purposes (see Lemma 5).

We remark that when $m=1$, things simplify dramatically, as we can simply integrate by parts (2) and get

$$\|\nabla u_k\|_{L^{(2,2)}(\Omega)} = \|\nabla u_k\|_{L^2(\Omega)} \le C.$$

In the case $m=2$, Robert and Struwe [22] proved a slightly weaker form of our Lemma 6 by using subtle estimates in the $BMO$ space, whose generalization to arbitrary dimensions appears quite challenging. Our approach, on the other hand, is simpler and more transparent.

Recently Druet [12] for the case $m=1$, and Struwe [27] for $m=2$ improved the previous results by showing that in case (ii) of Theorem 1 we have $\Lambda = L\Lambda_1$ for some positive $L \in \mathbb{N}$. Whether the same holds true for $m>2$ is still an open question. In is also unknown whether $L=I$ in case $m=1,2$.

In the following, the letter $C$ denotes a generic positive constant, which may change from line to line and even within the same line.

## 2 Proof of Theorem 1

Assume first that $\sup_\Omega u_k \le C$. Then $\Delta^m u_k \to 0$ uniformly, since $\lambda_k \to 0$. By elliptic estimates we infer $u_k \to 0$ in $W^{2m,p}(\Omega)$ for every $1 \le p < \infty$, hence $u_k \to 0$ in $C^{2m-1,\alpha}(\Omega)$, $\Lambda = 0$ and we are in case (i) of Theorem 1.

From now on, following the approach of [22], we assume that, up to a subsequence, $\sup_\Omega u_k \to \infty$ and show that we are in case (ii) of the theorem. In Sect. 2.1 we analyze the asymptotic profile at blow-up points. In Sect. 2.2 we sketch the inductive procedure which completes the proof.

### 2.1 Analysis of the first blow-up

Let $x_k = x_{1,k} \in \Omega$ be a point such that $u_k(x_k) = \max_\Omega u_k$, and let $r_k = r_{1,k}$ be as in (3). Integrating by parts in (2), we find $\|\Delta^{\frac{m}{2}} u_k\|_{L^2(\Omega)} \le C$ which, together with the boundary condition and elliptic estimates (see e.g. [4]), gives

$$\|u_k\|_{H^m(\Omega)} \le C. \tag{12}$$

**Lemma 2** *We have*

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_k, \partial\Omega)}{r_k} = +\infty.$$

*Proof* Set

$$\overline{u}_k(x) := \frac{u_k(r_k x + x_k)}{u_k(x_k)} \quad \text{for } x \in \Omega_k := \{r_k^{-1}(x - x_k) : x \in \Omega\}.$$

Then $\overline{u}_k$ satisfies

$$\begin{cases} (-\Delta)^m \overline{u}_k = \dfrac{2^{2m}(2m-1)!}{u_k^2(x_k)} \overline{u}_k e^{m u_k^2(x_k)(\overline{u}_k^2 - 1)} & \text{in } \Omega_k \\ \overline{u}_k > 0 & \text{in } \Omega_k \\ \overline{u}_k = \partial_\nu \overline{u}_k = \cdots = \partial_\nu^{m-1} \overline{u}_k = 0 & \text{on } \partial\Omega_k. \end{cases}$$

Assume for the sake of contradiction that up to a subsequence we have

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_k, \partial\Omega)}{r_k} = R_0 < +\infty.$$

Then, passing to a further subsequence, $\Omega_k \to \mathcal{P}$, where $\mathcal{P}$ is a half-space. Since

$$\|\Delta^m \overline{u}_k\|_{L^\infty(\Omega_k)} \le \frac{C}{u_k^2(x_k)} \to 0 \quad \text{as } k \to \infty,$$

we see that, up to a subsequence, $\overline{u}_k \to \overline{u}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\mathcal{P}})$, where

$$\overline{u}(0) = \overline{u}_k(0) = 1$$

and

$$\begin{cases} (-\Delta)^m \overline{u} = 0 & \text{in } \mathcal{P} \\ \overline{u} = \partial_\nu \overline{u} = \cdots = \partial_\nu^{m-1} \overline{u} = 0 & \text{on } \partial\mathcal{P}. \end{cases}$$

By (12) and the Sobolev imbedding $H^{m-1}(\Omega) \hookrightarrow L^{2m}(\Omega)$, we find

$$\int_{\Omega_k} |\nabla \overline{u}_k|^{2m} dx = \frac{1}{u_k(x_k)^{2m}} \int_{\Omega} |\nabla u_k|^{2m} dx \le \frac{C}{u_k(x_k)^{2m}} \to 0, \quad \text{as } k \to \infty.$$

Then $\nabla \overline{u} \equiv 0$, hence $\overline{u} \equiv const = 0$ thanks to the boundary condition. That contradicts $\overline{u}(0) = 1$. □

**Lemma 3** *We have*

$$u_k(x_k + r_k x) - u_k(x_k) \to 0 \quad \textit{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}) \textit{ as } k \to \infty. \tag{13}$$

*Proof* Set

$$v_k(x) := u_k(x_k + r_k x) - u_k(x_k), \quad x \in \Omega_k$$

Then $v_k$ solves

$$(-\Delta)^m v_k = 2^{2m}(2m-1)! \frac{\overline{u}_k(x)}{u_k(x_k)} e^{m u_k^2(x_k)(\overline{u}_k^2 - 1)} \le 2^{2m} \frac{(2m-1)!}{u_k(x_k)} \to 0. \tag{14}$$

Assume that $m > 1$. By (12) and the Sobolev embedding $H^{m-2}(\Omega) \hookrightarrow L^m(\Omega)$, we get

$$\|\nabla^2 v_k\|_{L^m(\Omega_k)} = \|\nabla^2 u_k\|_{L^m(\Omega)} \le C. \tag{15}$$

Fix now $R > 0$ and write $v_k = h_k + w_k$ on $B_R = B_R(0)$, where $\Delta^m h_k = 0$ and $w_k$ satisfies the Navier-boundary condition on $B_R$. Then, (14) gives

$$w_k \to 0 \quad \text{in } C^{2m-1,\alpha}(B_R). \tag{16}$$

This, together with (15) implies

$$\|\Delta h_k\|_{L^m(B_R)} \le C. \tag{17}$$

Then, since $\Delta^{m-1}(\Delta h_k) = 0$, we get from Proposition 12

$$\|\Delta h_k\|_{C^\ell(B_{R/2})} \le C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{18}$$

By Pizzetti's formula (45),

$$\fint_{B_R} h_k dx = h_k(0) + \sum_{i=1}^{m-1} c_i R^{2i} \Delta^i h_k(0),$$

and (18), together with $|h_k(0)| = |w_k(0)| \le C$ and $h_k \le -w_k \le C$, we find

$$\fint_{B_R} |h_k| dx \le C.$$

Again by Proposition 12 it follows that

$$\|h_k\|_{C^\ell(B_{R/2})} \le C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{19}$$

By Ascoli–Arzelà's theorem, (16) and (19), we have that up to a subsequence

$$v_k \to v \quad \text{in } C^{2m-1,\alpha}(B_{R/2}),$$

where $\Delta^m v \equiv 0$ thanks to (14). We can now apply the above procedure with a sequence of radii $R_k \to \infty$, extract a diagonal subsequence $(v_{k'})$, and find a function $v \in C^\infty(\mathbb{R}^{2m})$ such that

$$v \le 0, \quad \Delta^m v \equiv 0, \quad v_{k'} \to v \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}). \tag{20}$$

By Fatou's Lemma

$$\|\nabla^2 v\|_{L^m(\mathbb{R}^{2m})} \le \liminf_{k\to\infty} \|\nabla^2 v_{k'}\|_{L^m(\Omega_k)} \le C. \tag{21}$$

By Theorem 13 and (20), $v$ is a polynomial of degree at most $2m-2$. Then (20) and (21) imply that $v$ is constant, hence $v \equiv v(0) = 0$. Therefore the limit does not depend on the chosen subsequence $(v_{k'})$, and the full sequence $(v_k)$ converges to 0 in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, as claimed.

When $m = 1$, Pizzetti's formula and (14) imply at once that, for every $R > 0$, $\|v_k\|_{L^1(B_R)} \to 0$, hence $v_k \to 0$ in $W^{2,p}(B_{R/2})$ as $k \to \infty$, $1 \le p < \infty$. □

Now set

$$\eta_k(x) := u_k(x_k)[u_k(r_k x + x_k) - u_k(x_k)] + \log 2 \le \log 2. \tag{22}$$

An immediate consequence of Lemma 3 is the following

**Corollary 4** *The function $\eta_k$ satisfies*

$$(-\Delta)^m \eta_k = V_k e^{2m a_k \eta_k}, \tag{23}$$

*where*

$$V_k(x) = 2^{m(1-\overline{u}_k)}(2m-1)!\overline{u}_k(x) \to (2m-1)!, \quad a_k = \frac{1}{2}(\overline{u}_k + 1) \to 1$$

*in $C^0_{\mathrm{loc}}(\mathbb{R}^{2m})$.*

**Lemma 5** *For every $1 \le \ell \le 2m-1$, $\nabla^\ell u_k$ belongs to the Lorentz space $L^{(2m/\ell,2)}(\Omega)$ and*

$$\|\nabla^\ell u_k\|_{(2m/\ell,2)} \le C. \tag{24}$$

*Proof* We first show that $f_k := (-\Delta)^m u_k$ is bounded in $L(\log L)^{\frac{1}{2}}(\Omega)$, where

$$L(\log L)^\alpha(\Omega) := \left\{ f \in L^1(\Omega) : \|f\|_{L(\log L)^\alpha} := \int_\Omega |f| \log^\alpha(2+|f|)dx < \infty \right\}.$$

Indeed, set $\log^+ t := \max\{0, \log t\}$ for $t > 0$. Then, using the simple inequalities

$$\log(2+t) \le 2 + \log^+ t, \quad \log^+(ts) \le \log^+ t + \log^+ s, \quad t, s > 0,$$

one gets

$$\log(2 + \lambda_k u_k e^{m u_k^2}) \le 2 + \log^+ \lambda_k + \log^+ u_k + m u_k^2 \le C(1+u_k)^2.$$

Then, since $f_k \ge 0$, we have

$$\begin{aligned} \|f_k\|_{L(\log L)^{\frac{1}{2}}} &\le \int_\Omega f_k \log^{\frac{1}{2}}(2+f_k)dx \\ &\le C \int_{\{x\in\Omega: u_k(x)\ge 1\}} \lambda_k u_k^2 e^{m u_k} dx + C|\Omega| \le C \end{aligned}$$

by (2), as claimed. Now (24) follows from Theorem 10. □

*Remark* The inequality (24) is intermediate between the $L^1$ and the $L \log L$ estimates. Indeed, the bound of $f_k := (-\Delta)^m u_k$ in $L^1$ implies $\|\nabla^\ell u_k\|_{L^p} \le C$ for every $1 \le \ell \le 2m-1$, $1 \le p < \frac{2m}{\ell}$, and actually $\|\nabla^\ell u_k\|_{(2m/\ell,\infty)} \le C$ (compare [14, Thm. 3.3.6]), but that is not enough for our purposes (Lemma 6 below). On the other hand, was $f_k$ bounded in $L(\log L)$, we would have $\|\nabla^\ell u_k\|_{(2m/\ell,1)} \le C$, which implies $\|u_k\|_{L^\infty} \le C$ (compare [14, Thm. 3.3.8]). But we know that this is not the case in general.

Actually, the cases $1 \le \ell \le m$ in (24) follow already from (12) and the improved Sobolev embeddings, see [19]. What really matters here are the cases $m < \ell < 2m$. In fact, when $m = 1$ Lemma 5 reduces to (12).

The following lemma replaces and sharpens Proposition 2.3 in [22].

**Lemma 6** *For any $R > 0$, $1 \le \ell \le 2m-1$ there exists $k_0 = k_0(R)$ such that*

$$u_k(x_k) \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \le C(Rr_k)^{2m-\ell}, \quad \text{for all } k \ge k_0.$$

*Proof* We first claim that

$$\|\Delta^m(u_k^2)\|_{L^1(\Omega)} \le C. \tag{25}$$

To see that, observe that

$$|\Delta^m(u_k^2)| \le 2u_k(-\Delta)^m u_k + C \sum_{\ell=1}^{2m-1} |\nabla^\ell u_k||\nabla^{2m-\ell} u_k|. \tag{26}$$

The term $2u_k(-\Delta)^m u_k$ is bounded in $L^1$ thanks to (2). The other terms on the right-hand side of (26) are bounded in $L^1$ thanks to Lemma 5 and the Hölder-type inequality of O'Neil [19].[3] Hence (25) is proven.

[3] If $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, and $f \in L^{(p,q)}$, $g \in L^{(p',q')}$, then $\|fg\|_{L^1} \le \|f\|_{(p,q)} \|g\|_{(p',q')}$.

Now set $f_k := (-\Delta)^m(u_k^2)$, and for any $x \in \Omega$, let $G_x$ be the Green's function for $(-\Delta)^m$ on $\Omega$ with Dirichlet boundary condition. Then

$$u_k^2(x) = \int_\Omega G_x(y) f_k(y) dy.$$

Thanks to [11, Thm. 12], $|\nabla^\ell G_x(y)| \le C|x-y|^{-\ell}$, hence

$$|\nabla^\ell(u_k^2)(x)| \le \int_\Omega |\nabla_x^\ell G_x(y)||f_k(y)|dy \le C \int_\Omega \frac{|f_k(y)|}{|x-y|^\ell} dy.$$

Let $\mu_k$ denote the probability measure $\frac{|f_k(y)|}{\|f_k\|_{L^1(\Omega)}}dy$. By Fubini's theorem

$$\begin{aligned}\int_{B_{Rr_k}(x_k)} |\nabla^\ell(u_k^2)(x)|dx &\le C\|f_k\|_{L^1(\Omega)} \int_{B_{Rr_k}(x_k)} \int_\Omega \frac{1}{|x-y|^\ell} d\mu_k(y)dx \\ &\le C \int_\Omega \int_{B_{Rr_k}(x_k)} \frac{1}{|x-y|^\ell} dx d\mu_k(y) \\ &\le C \sup_{y\in\Omega} \int_{B_{Rr_k}(x_k)} \frac{1}{|x-y|^\ell} dx \le C(Rr_k)^{2m-\ell}.\end{aligned}$$

To conclude the proof, observe that Lemma 3 implies that on $B_{Rr_k}(x_k)$, for $1 \le \ell \le 2m-1$, we have $r_k^\ell \nabla^\ell u_k \to 0$ uniformly, hence

$$\begin{aligned}u_k(x_k)|\nabla^\ell u_k| \le C u_k |\nabla^\ell u_k| &\le C\left(|\nabla^\ell(u_k^2)| + \sum_{j=1}^{\ell-1} |\nabla^j u_k||\nabla^{\ell-j} u_k|\right) \\ &\le C|\nabla^\ell(u_k^2)| + o(r_k^{-\ell}), \quad \text{as } k \to \infty.\end{aligned}$$

Integrating over $B_{Rr_k}(x_k)$ and using the above estimates we conclude. □

**Proposition 7** *Let $\eta_k$ be as in (22). Then, up to selecting a subsequence, $\eta_k(x) \to \eta_0(x) = \log \frac{2}{1+|x|^2}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, and*

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_{Rr_k}(x_k)} \lambda_k u_k^2 e^{m u_k^2} dx = \lim_{R\to\infty} (2m-1)! \int_{B_R(0)} e^{2m\eta_0} dx = \Lambda_1. \tag{27}$$

*Proof* Fix $R > 0$, and notice that, thanks to Lemma 3 and (23),

$$\begin{aligned}\int_{B_R(0)} V_k e^{2m a_k \eta_k} dx &= \int_{B_{Rr_k}(x_k)} u_k(x_k) u_k \lambda_k e^{m u_k^2} dx \\ &\le (1+o(1)) \int_{B_{Rr_k}(x_k)} u_k^2 \lambda_k e^{m u_k^2} dx \le \Lambda + o(1),\end{aligned} \tag{28}$$

where $V_k$ and $a_k$ are as in Corollary 4, and $o(1) \to 0$ as $k \to \infty$.

*Step 1* We claim that $\eta_k \to \overline{\eta}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where $\overline{\eta}$ satisfies

$$(-\Delta)^m \overline{\eta} = (2m-1)! e^{2m\overline{\eta}}. \tag{29}$$

Then, letting $R \to \infty$ in (28), from Corollary 4 and Fatou's lemma we infer $e^{2m\overline{\eta}} \in L^1(\mathbb{R}^{2m})$.

Let us prove the claim. Consider first the case $m > 1$. From Corollary 4, Theorem 1 in [17], and (28), together with $\eta_k \leq \log 2$ (which implies that $S_1 = \emptyset$ in Theorem 1 of [17]), we infer that up to subsequences either

(i) $\eta_k \to \overline{\eta}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$ for some function $\overline{\eta} \in C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, or
(ii) $\eta_k \to -\infty$ locally uniformly in $\mathbb{R}^{2m}$, or
(iii) there exists a closed set $S_0 \neq \emptyset$ of Hausdorff dimension at most $2m-1$ and numbers $\beta_k \to +\infty$ such that

$$\frac{\eta_k}{\beta_k} \to \varphi \text{ in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0),$$

where

$$\Delta^m \varphi \equiv 0, \quad \varphi \leq 0, \quad \varphi \not\equiv 0 \quad \text{on } \mathbb{R}^{2m}, \quad \varphi \equiv 0 \text{ on } S_0. \tag{30}$$

Since $\eta_k(0) = \log 2$, (ii) can be ruled out. Assume now that (iii) occurs. From Liouville's theorem and (30) we get $\Delta\varphi \not\equiv 0$, hence for some $R > 0$ we have $\int_{B_R} |\Delta\varphi| dx > 0$ and

$$\lim_{k\to\infty} \int_{B_R} |\Delta\eta_k| dx = \lim_{k\to\infty} \beta_k \int_{B_R} |\Delta\varphi| dx = +\infty. \tag{31}$$

On the other hand, we infer from Lemma 6

$$\int_{B_R} |\nabla^\ell \eta_k| dx = u_k(x_k) r_k^{\ell-2m} \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \leq C R^{2m-\ell}, \tag{32}$$

contradicting (31) when $\ell = 2$ and therefore proving our claim.

When $m = 1$, Theorem 3 in [8] implies that only Case (i) or Case (ii) above can occur. Again Case (ii) can be ruled out, since $\eta_k(0) = \log 2$, and we are done.

*Step 2* We now prove that $\overline{\eta}$ is a standard solution of (29), i.e. there are $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$ such that

$$\overline{\eta}(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}. \tag{33}$$

For $m = 1$ this follows at once from [10]. For $m > 1$, if $\overline{\eta}$ didn't have the form (33), according to [16, Thm. 2] (see also [15] for the case $m = 2$), there would exist $j \in \mathbb{N}$ with $1 \leq j \leq m-1$, and $a < 0$ such that

$$\lim_{|x|\to\infty} (-\Delta)^j \overline{\eta}(x) = a.$$

This would imply

$$\lim_{k\to\infty} \int_{B_R(0)} |\Delta^j \eta_k| dx = |a| \cdot \mathrm{vol}(B_1(0)) R^{2m} + o(R^{2m}) \quad \text{as } R \to \infty,$$

contradicting (32) for $\ell = 2j$. Hence (33) is established. Since $\eta_k \leq \eta_k(0) = \log 2$, it follows immediately that $x_0 = 0$, $\lambda = 1$, i.e. $\overline{\eta} = \eta_0$, and (27) follows from (11), (28) and Fatou's lemma. □

### 2.2 Exhaustion of the blow-up points and proof of Theorem 1

For $\ell \in \mathbb{N}$ we say that $(H_\ell)$ holds if there are $\ell$ sequences of converging points $x_{i,k} \to x^{(i)}$, $1 \le i \le \ell$ such that

$$\sup_{x\in\Omega} \lambda_k R_{\ell,k}^{2m}(x) u_k^2(x) e^{mu_k^2(x)} \le C, \tag{34}$$

where

$$R_{\ell,k}(x) := \inf_{1\le i\le \ell} |x - x_{i,k}|.$$

We say that $(E_\ell)$ holds if there are $\ell$ sequences of converging points $x_{i,k} \to x^{(i)}$ such that, if we define $r_{i,k}$ as in (3), the following hold true:

$(E_\ell^1)$ For all $1 \le i \ne j \le \ell$

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_{i,k}, \partial\Omega)}{r_{i,k}} = \infty, \qquad \lim_{k\to\infty} \frac{|x_{i,k} - x_{j,k}|}{r_{i,k}} = \infty.$$

$(E_\ell^2)$ For $1 \le i \le \ell$ (4) holds true.

$(E_\ell^3)$ $\lim_{R\to\infty} \lim_{k\to\infty} \int_{\cup_{i=1}^{\ell} B_{Rr_{i,k}}(x_{i,k})} \lambda_k u_k^2 e^{mu_k^2} dx = \ell \Lambda_1.$

To prove Theorem 1 we show inductively that $(H_I)$ and $(E_I)$ hold for some positive $I \in \mathbb{N}$ (with the same sequences $x_{i,k} \to x^{(i)}$, $1 \le i \le I$), following the approach of [2,22]. First observe that $(E_1)$ holds thanks to Lemma 2 and Proposition 7. Assume now that for some $\ell \ge 1$ $(E_\ell)$ holds and $(H_\ell)$ does not. Choose $x_{\ell+1,k} \in \Omega$ such that

$$\lambda_k R_{\ell,k}^{2m}(x_{\ell+1,k}) u_k^2(x_{\ell+1,k}) e^{mu_k^2(x_{\ell+1,k})} = \lambda_k \max_{\Omega} R_{\ell,k}^{2m} u_k^2 e^{mu_k^2} \to \infty \quad \text{as } k \to \infty \tag{35}$$

and define $r_{\ell+1,k}$ as in (3). It easily follows from (35) that

$$\lim_{k\to\infty} \frac{|x_{\ell+1,k} - x_{i,k}|}{r_{\ell+1,k}} = \infty, \quad 1 \le i \le \ell. \tag{36}$$

Moreover, thanks to $(E_\ell^2)$ and (35), we also have

$$\lim_{k\to\infty} \frac{|x_{\ell+1,k} - x_{i,k}|}{r_{i,k}} = \infty \quad \text{for } 1 \le i \le \ell.$$

We now need to replace Lemmas 2 and 3 with the lemma below.

**Lemma 8** *Under the above assumptions and notation, we have*

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_{\ell+1,k}, \partial\Omega)}{r_{\ell+1,k}} = \infty \tag{37}$$

*and*

$$u_k(x_{\ell+1,k} + r_{\ell+1,k} x) - u_k(x_{\ell+1,k}) \to 0 \;\; \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}), \quad \text{as } k \to \infty. \tag{38}$$

*Proof* To simplify the notation, let us write $y_k := x_{\ell+1,k}$ and $\rho_k := r_{\ell+1,k}$. Evaluating the right-hand side of (35) at the point $y_k + \rho_k x$ we get

$$\begin{aligned} &\left(\inf_{1\le i\le \ell} |y_k - x_{i,k} + \rho_k x|^{2m}\right) u_k^2(y_k + \rho_k x) e^{mu_k^2(y_k+\rho_k x)} \\ &\le \left(\inf_{1\le i\le \ell} |y_k - x_{i,k}|^{2m}\right) u_k^2(y_k) e^{mu_k^2(y_k)}, \end{aligned}$$

Hence, setting $\overline{u}_{\ell+1,k}(x) := \frac{u_k(y_k+\rho_k x)}{u_k(y_k)}$, we have that

$$\overline{u}^2_{\ell+1,k}(x)e^{mu_k^2(y_k)(\overline{u}^2_{\ell+1,k}(x)-1)} \le \frac{\inf_{1\le i\le \ell}|y_k - x_{i,k}|^{2m}}{\inf_{1\le i\le \ell}|y_k - x_{i,k}+\rho_k x|^{2m}} = 1+o(1), \tag{39}$$

where $o(1)\to 0$ as $k\to\infty$ locally uniformly in $x$, as (36) immediately implies. Then (37) follows as in the proof of Lemma 2, since (39) implies

$$(-\Delta)^m\overline{u}_{\ell+1,k} = \frac{2^{2m}(2m-1)!}{u_k^2(y_k)}\overline{u}_{\ell+1,k}e^{mu_k^2(y_k)(\overline{u}^2_{\ell+1,k}-1)} = o(1), \tag{40}$$

where $o(1)\to 0$ as $k\to\infty$ uniformly locally in $\mathbb{R}^{2m}$.

Define now $v_k(x) := u_k(x_{\ell+1,k}+r_{\ell+1,k}x) - u_k(x_{\ell+1,k})$, and observe that

$$u_k(y_k+\rho_k x)\to\infty \quad \text{locally uniformly in } \mathbb{R}^{2m},$$

thanks to (35) and (36). This and (40) imply that we can replace (14) in the proof of Lemma 3 with

$$(-\Delta)^m v_k = 2^{2m}(2m-1)!\frac{\overline{u}_k^2}{u_k(y_k+\rho_k\cdot)}e^{mu_k^2(y_k)(\overline{u}^2_{\ell+1,k}-1)} \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^{2m}).$$

Then the rest of the proof of Lemma 3 applies without changes, and also (38) is proved. □

Still repeating the arguments of the preceding section with $x_{\ell+1,k}$ instead of $x_k$ and $r_{\ell+1,k}$ instead of $r_k$, we define

$$\eta_{\ell+1,k}(x) := u_k(x_{\ell+1,k})[u_k(r_{\ell+1,k}x + x_{\ell+1,k}) - u_k(x_{\ell+1,k})],$$

and we have

**Proposition 9** *Up to a subsequence*

$$\eta_{\ell+1,k}(x)\to\eta_0(x) = \log\frac{2}{1+|x|^2} \quad \textit{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$$

*and*

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{Rr_{\ell+1,k}}(x_{\ell+1,k})}\lambda_k u_k^2 e^{mu_k^2}dx = \lim_{R\to\infty}\int_{B_R(0)} e^{2m\eta_0}dx = \Lambda_1. \tag{41}$$

Summarizing, we have proved that $(E^1_{\ell+1})$, $(E^2_{\ell+1})$ and (41) hold. These also imply that $(E^3_{\ell+1})$ holds, hence we have $(E_{\ell+1})$. Because of (2) and $(E^3_\ell)$, the procedure stops in a finite number $I$ of steps, and we have $(H_I)$.

Finally, we claim that $\lambda_k\to 0$ implies $u_k \rightharpoonup 0$ in $H^m(\Omega)$. This, (5) and elliptic estimates then imply that

$$u_k\to 0 \quad \text{in} \quad C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash\{x^{(1)},\dots,x^{(I)}\}).$$

To prove the claim, we observe that for any $\alpha > 0$

$$\begin{aligned}\int_\Omega |\Delta^m u_k| dx &= \int_\Omega \lambda_k u_k e^{m u_k^2} dx \\ &\le \frac{\lambda_k}{\alpha} \int_{\{x\in\Omega: u_k \ge \alpha\}} u_k^2 e^{m u_k^2} dx + \lambda_k \int_{\{x\in\Omega: u_k < \alpha\}} u_k e^{m u_k^2} dx \\ &\le \frac{C}{\alpha} + \lambda_k C_\alpha,\end{aligned}$$

where $C_\alpha$ depends only on $\alpha$. Letting $k$ and $\alpha$ go to infinity, we infer

$$\Delta^m u_k \to 0 \quad \text{in } L^1(\Omega). \tag{42}$$

Thanks to (12), we infer that up to a subsequence $u_k \rightharpoonup u_0$ in $H^m(\Omega)$. Then (42) and the boundary condition imply that $u_0 \equiv 0$, in particular the full sequence converges to 0 weakly in $H^m(\Omega)$. This completes the proof of the theorem.

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## Appendix

### An elliptic estimate for Zygmund and Lorentz spaces

**Theorem 10** *Let $u$ solve $\Delta^m u = f \in L(\log L)^\alpha$ in $\Omega$ with Dirichlet boundary condition, $0 \le \alpha \le 1$, $\Omega \subset \mathbb{R}^n$ bounded and with smooth boundary, $n \ge 2m$. Then $\nabla^{2m-\ell} u \in L^{\left(\frac{n}{n-\ell}, \frac{1}{\alpha}\right)}(\Omega)$, $1 \le \ell \le 2m-1$ and*

$$\|\nabla^{2m-\ell} u\|_{\left(\frac{n}{n-\ell}, \frac{1}{\alpha}\right)} \le C \|f\|_{L(\log L)^\alpha}. \tag{43}$$

*Proof* Define

$$\hat{f} := \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

and let $w := K * \hat{f}$, where $K$ is the fundamental solution of $\Delta^m$. Then

$$|\nabla^{2m-1} w| = |(\nabla^{2m-1} K) * \hat{f}| \le C I_1 * |\hat{f}|,$$

where $I_1(x) = |x|^{1-n}$. According to [5, Cor. 6.16], $|\nabla^{2m-1} w| \in L^{\left(\frac{n}{n-1}, \frac{1}{\alpha}\right)}(\mathbb{R}^n)$ and

$$\|\nabla^{2m-1} w\|_{\left(\frac{n}{n-1}, \frac{1}{\alpha}\right)} \le C \|\hat{f}\|_{L(\log L)^\alpha} = C \|f\|_{L(\log L)^\alpha}. \tag{44}$$

We now use (44) to prove (43), following a method that we learned from [14]. Given $g : \Omega \to \mathbb{R}^n$ measurable, let $v_g$ be the solution to $\Delta^m v_g = \operatorname{div} g$ in $\Omega$, with the same boundary condition as $u$, and set $P(g) := |\nabla^{2m-1} v_g|$. By $L^p$ estimates (see e.g. [4]), $P$ is bounded from $L^p(\Omega; \mathbb{R}^n)$ into $L^p(\Omega)$ for $1 < p < \infty$. Then, thanks to the interpolation theory for Lorentz spaces, see e.g. [14, Thm. 3.3.3], $P$ is bounded from $L^{(p,q)}(\Omega; \mathbb{R}^n)$ into $L^{(p,q)}(\Omega)$ for $1 < p < \infty$ and $1 \le q \le \infty$. Choosing now $g = \nabla \Delta^{m-1} w$, we get $v_g = u$, hence

$|\nabla^{2m-1}u| = P(\nabla\Delta^{m-1}w)$, and from (44) we infer

$$\|\nabla^{2m-1}u\|_{\left(\frac{n}{n-1},\frac{1}{\alpha}\right)} \leq C\|\nabla\Delta^{m-1}w\|_{\left(\frac{n}{n-1},\frac{1}{\alpha}\right)} \leq C\|f\|_{L(\log L)^\alpha}.$$

For $1 < \ell \leq 2m-1$ (43) follows from the Sobolev embeddings, see [19]. □

### Other useful results

A proof of the results below can be found in [16]. The following Lemma can be considered a generalized mean value identity for polyharmonic function.

**Lemma 11** (Pizzetti [21]) *Let $u \in C^{2m}(B_R(x_0))$, $B_R(x_0) \subset \mathbb{R}^n$, for some $m, n$ positive integers. Then there are positive constants $c_i = c_i(n)$ such that*

$$\fint_{B_R(x_0)} u(x)dx = \sum_{i=0}^{m-1} c_i R^{2i}\Delta^i u(x_0) + c_m R^{2m}\Delta^m u(\xi), \tag{45}$$

*for some $\xi \in B_R(x_0)$.*

**Proposition 12** *Let $\Delta^m h = 0$ in $B_2 \subset \mathbb{R}^n$. For every $0 \leq \alpha < 1$, $p \in [1,\infty)$ and $\ell \geq 0$ there are constants $C(\ell, p)$ and $C(\ell,\alpha)$ independent of $h$ such that*

$$\|h\|_{W^{\ell,p}(B_1)} \leq C(\ell,p)\|h\|_{L^1(B_2)}$$
$$\|h\|_{C^{\ell,\alpha}(B_1)} \leq C(\ell,\alpha)\|h\|_{L^1(B_2)}.$$

A simple consequence of Lemma 11 and Proposition 12 is the following Liouville-type Theorem.

**Theorem 13** *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h(x) \leq C(1+|x|^\ell)$ for some $\ell \geq 0$. Then $h$ is a polynomial of degree at most* $\max\{\ell, 2m-2\}$.

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